Ecuaţii diferenţiale de ordin superior

$$F(x, y, y', ..., y^{(n)}) = 0, \qquad (1)$$

unde $F : G \subseteq \mathbb{R}^{n+2} \to \mathbb{R}$.

$$y^{(n)} = f(x, y, y', ..., y^{(n-1)}), \qquad (2)$$

unde $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$.

O functie $\varphi(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R}$, unde I este un interval in $\mathbb{R}$, este o solutie a ecuatiei (1) daca $\varphi(\cdot)$ este de n-ori derivabila, $(x, \varphi(x), \varphi'(x), \varphi''(x), ..., \varphi^{(n-1)}(x)) \in D$, $\forall x \in I$ si $\varphi(\cdot)$ verifica ecuatia:

$$\varphi^{(n)}(x) = f(x, \varphi(x), ..., \varphi^{(n-1)}(x)), \quad x \in I.$$

Problema Cauchy:

$$\begin{cases} y^{(n)}(x) = f(x, y, y^{'}, ..., y^{(n-1)}), \\ \\ y(x_0) = y_0,\ y^{'}(x_0) = y_0^{'},\ \dots, y^{(n-1)}(x_0) = y_0^{(n-1)}. \end{cases} \tag{3}$$

Solutia generala a ecuatiei (1) depinde de n parametri $C_1, C_2, ..., C_n$:

$$y = y(x, C_1, C_2, ..., C_n).$$

Solutia generala sub forma implicita:

$$\Phi(x, y, C_1, C_2, ..., C_n) = 0.$$

Caz particular.

$$y^{(n)}(x) = f(x), \tag{4}$$

cu f continua pe intervalul $I \subseteq \mathbb{R}$.

Solutia generala:

$$y(x) = \frac{1}{(n-1)!} \int_{x_0}^{x} (x-t)^{n-1} f(t) dt + P_{n-1}(x - x_0), \ \forall x, x_0 \in I, \quad (5)$$

unde $P_{n-1}(x - x_0)$ este un polinom de gradul $(n-1)$ in $(x - x_0)$ cu coeficienti constanti arbitrari.

Exemplul 1.

$$y^{'''} = 12x. \quad \blacksquare$$

$$y(x) = \frac{x^4}{2} + C_1 \frac{x^2}{2} + C_2 x + C_3, \quad C_1, C_2, C_3 \in \mathbb{R}.$$

Exemplul 2.

Integrati ecuatia:

$$y'' = \sin x. \quad \blacksquare$$

Integrand de doua ori, obtinem

$$y(x) = -\sin x + C_1 x + C_2, \quad C_1, C_2 \in \mathbb{R}.$$

Exemplul 3. Determinati legea de miscare a unui punct material de masa m, aruncat vertical in sus cu o viteza initiala v_0, presupunand ca rezistenta aerului poate fi neglijata. $\blacksquare$

Luam verticala ca axa Ox. Din legea lui Newton, avem:

$$m\frac{d^2x}{dt^2} = -mg. \tag{6}$$

$$\begin{cases} \dfrac{d^2x}{dt^2} = -g, \\ \\ x(0) = 0, \ \dfrac{dx}{dt}(0) = v_0. \end{cases} \tag{7}$$

$$x(t) = v_0 t - \frac{gt^2}{2}. \tag{8}$$

Reducerea ordinului

$$F(x, y, y^{'}, ..., y^{(n)}) = 0.$$

I. Cazul in care nu apar explicit derivatele pana la ordinul $(k-1)$ inclusiv:

$$F(x, y^{(k)}, y^{(k+1)}, ..., y^{(n)}) = 0, \quad 1 \leq k \leq n. \tag{9}$$

Schimbarea de variabila:

$$z(x) = y^{(k)}(x) \tag{10}$$

ne conduce la

$$F(x, z, z^{'}, ..., z^{(n-k)}) = 0. \tag{11}$$

Integrand, rezulta:

$$z(x) = \psi(x, C_1, C_2, ..., C_{n-k}), \tag{12}$$

$$y^{(k)}(x) = \psi(x, C_1, C_2, ..., C_{n-k})$$

Integrand de k ori, obtinem solutia generala:

$$y(x) = \varphi(x, C_1, C_2, ..., C_n) \tag{13}$$

depinzand de n constante $C_1, C_2, ..., C_n$.

Sigur, putem avea si solutii singulare z_i, care, prin integrare de k ori, ne conduc la solutiile singulare y_i.

Exemplul 1.

Aflati solutia problemei Cauchy:

$$\begin{cases} xy'' + y' = -x^2 y'^2, \\ \\ y(1) = y'(1) = 1. \quad \blacksquare \end{cases}$$

$$z(x) = y'(x).$$

$$\begin{cases} xz' + z = -x^2 z^2, \\ \\ z(1) = 1. \end{cases}$$

Integrand ecuatia Bernoulli, obtinem

$$z(x) = \frac{1}{x^2}.$$

Astfel, solutia problemei initiale este:

$$y(x) = -\frac{1}{x} + 2.$$

Exemplul 2.

Integrati ecuatia:

$$xy^{'''} + y^{''} = 1 + x, \quad x \in I \subseteq R^*. \quad \blacksquare$$

$$z(x) = y^{''}(x).$$

$$\frac{dz}{dx} = -\frac{z}{x} + \frac{1+x}{x}.$$

Solutia generala

$$z(x) = \frac{x}{2} + 1 + \frac{C_1}{x}, \quad C_1 \in \mathbb{R}.$$

Deci:

$$y(x) = \frac{x^3}{12} + \frac{x^2}{2} + K_1 x \ln \mid x \mid + K_2 x + K_3, \quad K_1, K_2, K_3 \in \mathbb{R}.$$

Cazul II.

$$F(x, \frac{y^{'}}{y}, \frac{y^{''}}{y}, ..., \frac{y^{(n)}}{y}) = 0, \quad F : D \subseteq \mathbb{R}^{n+1} \to \mathbb{R}. \tag{14}$$

Schimbarea de variabila:

$$z(x) = \frac{y^{'}(x)}{y(x)}. \tag{15}$$

$$\Phi(x, z, z^{'}, ..., z^{(n-1)}) = 0. \tag{16}$$

Integrand, obtinem

$$z(x) = \psi(x, C_1, C_2, ..., C_{n-1}). \tag{17}$$

$$y^{'}(x) = y\psi(x, C_1, C_2, ..., C_{n-1}).$$

Solutia generala a ecuatiei initiale:

$$y(x) = \varphi(x, C_1, C_2, ..., C_n) \tag{18}$$

depinzand de n constante $C_1, C_2, ..., C_n$.

Sigur, putem avea si solutii singulare z_i, care, prin integrare, ne conduc la solutiile singulare y_i.

Exemplul 1.

Aflati solutia problemei Cauchy:

$$\begin{cases} 2yy^{''} - 3y^{'2} - 4y^2 = 0, \\ \\ y(0) = 1,\ y^{'}(0) = 0,\ x \in (-\frac{\pi}{2}, \frac{\pi}{2}). \end{cases} \quad \blacksquare$$

$$z(x) = \frac{y^{'}(x)}{y(x)}.$$

$$\begin{cases} 2z^{'} = z^2 + 4, \\ \\ z(0) = 0. \end{cases}$$

$$z(x) = 2\tan\, x.$$

$$y(x) = \frac{1}{\cos^2 x}.$$

Exemplul 2.

Integrati ecuatia:

$$yy^{''} - y^{'2} = x^2y^2. \quad \blacksquare$$

Evident, $y = 0$ este solutie. Pentru $y \neq 0$, avem

$$\frac{y^{''}}{y} - (\frac{y^{'}}{y})^2 = x^2.$$

Schimbarea

$$z(x) = \frac{y^{'}(x)}{y(x)}$$

ne conduce la

$$z^{'} = x^2,$$

cu solutia generala

$$z(x) = \frac{x^3}{3} + C_1, \quad C_1 \in \mathbb{R}.$$

$$y' = y\,(\frac{x^3}{3} + C_1).$$

Deci:

$$y(x) = C_2\, e^{\frac{x^4}{12} + C_1 x}, \quad C_1 \in \mathbb{R},\ C_2 \in \mathbb{R}^*.$$

Cazul III. Cazul in care membrul stang nu depinde explicit de x:

$$F(y, y', y'', ..., y^{(n)}) = 0, \quad F : D \subseteq \mathbb{R}^{n+1} \to \mathbb{R}. \tag{19}$$

Schimbarea de variabila:

$$z(y(x)) = y'(x). \tag{20}$$

Obtinem

$$y'' = z' z, \quad y''' = z'' z^2 + z z'^2.$$

$$y^{(n)} = \varphi(z, z^{'}, ..., z^{(n-1)})$$

$$\Phi(y, z, z^{'}, ..., z^{(n-1)}) = 0. \tag{21}$$

Integrand, obtinem

$$z(y) = \psi(y, C_1, C_2, ..., C_{n-1}), \tag{22}$$

$$y(x) = \varphi(x, C_1, C_2, ..., C_n). \tag{23}$$

Sigur, putem avea si solutii singulare z_i, care, prin integrare, ne conduc la solutiile singulare y_i.

Exemplu.

$$\begin{cases} 2yy^{'} = y^{''}, \\ y(0) = 0,\ y^{'}(0) = 1,\ x \in (-\frac{\pi}{2}, \frac{\pi}{2}). \end{cases} \quad \blacksquare$$

$$z(y) = y^{'}(x)$$

$$\begin{cases} z \, \dfrac{dz}{dy} = 2yz, \\ \\ z(0) = 1. \end{cases}$$

$$z(y) = y^2 + 1.$$

Integram ecuatia

$$\frac{dy}{dx} = y^2 + 1.$$

Obtinem

$$y(x) = \tan{(x + C)}.$$

Din $y(0) = 0$ si $x \in (-\dfrac{\pi}{2}, \dfrac{\pi}{2})$, rezulta

$$y(x) = \tan\, x.$$

Cazul IV. Ecuatii Euler:

$$F(y, xy', x^2y'', ..., x^ny^{(n)}) = 0, \quad F : D \subseteq \mathbb{R}^{n+1} \to \mathbb{R},\ x \in I \subseteq \mathbb{R}^*. \tag{24}$$

Schimbarea de variabile:

$$| x |= e^s. \tag{25}$$

Vom considera $x > 0$.

$$z(s) = y(e^s),\ s = \ln\ x. \tag{26}$$

Astfel,

$$y' = \frac{z'}{x},$$

adica

$$xy' = z'.$$

Similar,

$$x^2y'' = z'' - z'.$$

Prin inductie,

$$x^n y^{(n)} = \varphi(z, z^{'}, z^{''}, ..., z^{(n)}).$$

Astfel, ecuatia (...) devine

$$\Phi(z, z^{'}, ..., z^{(n)}) = 0. \tag{27}$$

$$z(s) = \psi(s, C_1, C_2, ..., C_n). \tag{28}$$

$$y(x) = \psi(\ln\, x, C_1, C_2, ..., C_n). \tag{29}$$

Putem avea si solutii singulare z_i, care, prin integrare, ne conduc la solutiile singulare y_i.

Exemplu.

$$\begin{cases} x^2 y^{''} + x y^{'} + y = 0, \\ \\ y(1) = y^{'}(1) = 1,\, x > 0. \quad \blacksquare \end{cases}$$

Schimbarea de variabile:

$$x = e^s.$$

Obtinem

$$\begin{cases} z^{''} + z = 0, \\ \\ z(0) = 1,\, z^{'}(0) = 1. \end{cases}$$

$$z(s) = \cos\, s + \sin\, s.$$

Deci:

$$y(x) = \cos(\ln\, x) + \sin(\ln\, x).$$

Ecuatii liniare

$$a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + ... + a_{n-1}(x)y^{'}(x) + a_n(x)y(x) =$$

$$= f(x), \quad x \in (a, b) \tag{30}$$

unde $f, a_0, ..., a_n$ sunt functii continue pe intervalul (a, b).

Daca $f(x) \equiv 0$, ecuatia (30) s.n. *omogena*.

$$a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + ... + a_{n-1}(x)y^{'}(x) + a_n(x)y(x) =$$

$$= 0, \quad x \in (a, b). \tag{31}$$

Daca $a_0(x) \neq 0$,

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + ... + p_{n-1}(x)y^{'}(x) + p_n(x)y(x) =$$

$$= 0, \quad x \in (a, b). \qquad (32)$$

Conditii initiale:

$$y(x_0) = y_0,\ y^{'}(x_0) = y_0^{'}, \ldots, y^{(n-1)}(x_0) = y_0^{(n-1)},$$

pentru $x_0 \in (a, b)$.

Functiile $y_1(x)$,...,$y_n(x)$ s.n. *liniar dependente* pe (a, b) daca exista n constante $\alpha_1, \ldots, \alpha_n$, nu toate nule. a.i.

$$\alpha_1 y_1(x) + \ldots + \alpha_n y_n(x) \equiv 0, \quad x \in (a, b). \qquad (33)$$

Daca identitatea (33) este satisfacuta doar pentru $\alpha_1 = \ldots = \alpha_n = 0$, atunci functiile $y_1(x), \ldots, y_n(x)$ s.n. *liniar independente* pe intervalul (a, b). ■

Exemple.

(1) Functiile $e^{k_1 x}, e^{k_2 x}, ..., e^{k_n x},\ k_i \neq k_j, \forall i \neq j,\ k_i \in \mathbb{R}$, sunt liniar independente pe (a, b).

(2) Functiile $e^{kx}, xe^{kx}, x^2 e^{kx}, ..., x^p e^{kx}, \quad k \in \mathbb{R}, \quad p \in \mathbb{N}$, sunt liniar independente pe intervalul (a, b).

Daca $y_1(x), y_2(x), ..., y_n(x)$ sunt liniar dependente pe (a, b), atunci determinantul

$$W(x) \equiv W[y_1, ..., y_n] = \begin{vmatrix} y_1(x) & y_2(x) & \ldots\ldots\ldots\ldots & y_n(x) \\ y_1'(x) & y_2'(x) & \ldots\ldots\ldots\ldots & y_n'(x) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots\ldots & y_n^{(n-1)}(x) \end{vmatrix}$$

numit Wronskian, este identic zero. ■

Daca functiile liniar independente $y_1, ..., y_n$ sunt solutii ale ecuatiei omogene (32), atunci $W(x) = W[y_1, ..., y_n] \neq 0$ pentru orice $x \in (a, b)$.

■

Un ansamblu de n solutii liniar independente $y_1, ..., y_n$ ale ecuatiei (32) s.n. *sistem fundamental de solutii* ale acestei ecuatii.

Solutia generala a ecuatiei

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + ... + p_n(x)y(x) = 0 \tag{34}$$

este combinatia liniara

$$y(x) = C_1y_1(x) + ... + C_ny_n(x) \tag{35}$$

a n solutii liniar independente $y_1, ..., y_n$ on (a, b), cu n coeficienti arbitrari $C_1, ..., C_n$. ■

Numarul maxim de solutii liniar independente ale unei ecuatii liniare omogene cu coeficienti continui este egal cu ordinul ei. ∎

Fie ecuatia neomogena:

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + ... + p_n(x)y(x) = f(x), \quad x \in (a, b). \tag{36}$$

Solutia generala:

$$y(x) = \sum_{i=1}^{n} C_i y_i(x) + y_p(x). \quad \blacksquare \tag{37}$$

Metoda variatiei constantelor:

$$y(x) = C_1 y_1(x) + ... + C_n y_n(x), \tag{38}$$

unde $C_1(x), ..., C_n(x)$ sunt functii de clasa C^1 care urmeaza a fi determinate.

$$
\begin{cases}
C_1'(x)y_1(x) + C_2'(x)y_2(x) + ... + C_n'(x)y_n(x) = 0, \\
C_1'(x)y_1'(x) + C_2'(x)y_2'(x) + ... + C_n'(x)y_n'(x) = 0 \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
C_1'(x)y_1^{(n-2)}(x) + C_2'(x)y_2^{(n-2)}(x) + ... + C_n'(x)y_n^{(n-2)}(x) = 0.
\end{cases}
$$

$$
\begin{cases}
C_1'(x)y_1(x) + C_2'(x)y_2(x) + ... + C_n'(x)y_n(x) = 0, \\
C_1'(x)y_1'(x) + C_2'(x)y_2'(x) + ... + C_n'(x)y_n'(x) = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
C_1'(x)y_1^{(n-1)}(x) + C_2'(x)y_2^{(n-1)}(x) + ... + C_n'(x)y_n^{(n-1)}(x) = f(x).
\end{cases}
$$

$$\begin{vmatrix} y_1(x) & y_2(x) & \dots\dots\dots\dots\dots & y_n(x) \\ y_1^{'}(x) & y_2^{'}(x) & \dots\dots\dots\dots\dots & y_n^{'}(x) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & & & \\ y_1^{(n-1)}(x) & y_2^{'}(x) & \dots\dots & y_n^{(n-1)}(x) \end{vmatrix},$$

$$C_i^{'}(x) = \varphi_i(x), \quad i = 1, 2, ..., n,$$

unde φ_i sunt functii continue pe (a, b).

$$C_i(x) = \int \varphi_i(x)dx + C_i, \quad C_i \in \mathbb{R}, \quad i = 1, 2, ..., n.$$

$$y(x) = (\int \varphi_1(x)dx + C_1)y_1(x) + (\int \varphi_2(x)dx + C_2)y_2(x) + ...+$$

$$+(\int \varphi_n(x)dx + C_n)y_n(x), \quad C_1, ..., C_n \in \mathbb{R}. \tag{39}$$

Exemplu.

Integrati ecuatia

$$x^2y^{''} - 3xy^{'} - 5y = x^2, \quad x > 0. \quad \blacksquare \tag{40}$$

Ecuatia omogena

$$x^2y^{''} - 3xy^{'} - 5y = 0$$

este o ecuatie Euler. Efectuand schimbarea de variabile $x = e^s$, obtinem

$$y(x) = C_1x^5 + \frac{C_2}{x}.$$

Cautam solutia generala a ecuatiei neomogene de forma

$$y(x) = C_1(x)x^5 + \frac{C_2(x)}{x}.$$

Rezulta

$$\begin{cases} C_1'(x)x^5 + \dfrac{C_2'(x)}{x} = 0, \\ \\ 5C_1'(x)x^4 - \dfrac{C_2'(x)}{x^2} = x^2. \end{cases}$$

$$\begin{cases} C_1'(x) = \dfrac{1}{6x^2}, \\ \\ C_2'(x) = -\dfrac{x^4}{6}. \end{cases}$$

Integrand,

$$\begin{cases} C_1(x) = -\dfrac{1}{6x} + K_1, \\ \\ C_2 = -\dfrac{x^5}{30} + K_2, \quad K_1, K_2 \in \mathbb{R}. \end{cases}$$

Deci:

$$y(x) = (-\frac{1}{6x} + K_1)x^5 + (-\frac{x^5}{30} + K_2)\frac{1}{x}, \quad K_1, K_2 \in \mathbb{R}.$$

Ecuatii liniare cu coeficienti constanti

$$a_0 y^{(n)}(x) + a_1 y^{(n-1)}(x) + ... + a_{n-1} y^{'}(x) + a_n y(x) =$$
$$= f(x), \quad x \in (a, b)$$

unde f este o functie continua pe intervalul (a, b) si a_i, $i = 0, 1, ..., n$ sunt constante reale date.

Cazul omogen:

Cautam solutii de forma $y = e^{rx}$. Avem:

$$y' = re^{rx},\ y'' = r^2 e^{rx},\ ...,\ y^{(n)} = r^n e^{rx}.$$

Rezulta

$$e^{rx}(a_0 r^n + a_1 r^{n-1} + ... + a_{n-1} r + a_n) = 0.$$

Ecuatia caracteristica:

$$a_0 r^n + a_1 r^{n-1} + ... + a_{n-1} r + a_n = 0. \tag{41}$$

Ecuatia (41) are n radacini, numite *valori (radacini) caracteristice*.

Cazul 1. Daca radacinile $r_1, r_2, ..., r_n$ sunt reale si distincte, atunci

$$y_1 = e^{r_1 x},\ y_2 = e^{r_2 x}, ..., y_n = e^{r_n x}$$

formeaza un sistem fundamental de solutii.

Solutia generala a ecuatiei omogene atasate ecuatiei date este:

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} + ... + C_n e^{r_n x},$$

unde $C_1, C_2, ..., C_n$ sunt constante reale arbitrare.

Exemplu. Ecuatia

$$y^{''} + 5y^{'} + 6y = 0$$

are ecuatia caracteristica

$$r^2 + 5r + 6 = 0,$$

cu radacinile

$$r_1 = -2, \; r_2 = -3.$$

Deci,

$$y = C_1 e^{-2x} + C_2 e^{-3x}.$$

Cazul 2. Daca ecuatia caracteristica are radacinile reale r_i, cu multiplicitatile α_i, adica

$$P(r) = a_0(r - r_1)^{\alpha_1} \cdots (r - r_k)^{\alpha_k},$$

cu

$$\alpha_1 + ... + \alpha_k = n,$$

atunci

$$e^{r_1 x}, xe^{r_1 x}, ..., x^{\alpha_1 - 1}e^{r_1 x}, ..., e^{r_k x}, xe^{r_k x}, ..., x^{\alpha_k - 1}e^{r_k x}$$

formeaza un sistem fundamental de solutii si solutia generala a ecuatiei omogene este combinatia lor liniara cu n constante reale arbitrare $C_i,\ i = 1, ..., n$.

Exemplu. Ecuatia

$$y^{'''} + 3y^{''} + 3y^{'} + y = 0$$

are ecuatia caracteristica

$$r^3 + 3r^2 + 3r + 1 = 0,$$

adica

$$(r + 1)^3 = 0,$$

cu radacina tripla $r = -1$.

Solutia generala:

$$y = e^{-x}(C_1 + C_2 x + C_3 x^2), \quad C_i \in \mathbb{R}, \ i = 1, 2, 3.$$

Cazul 3. Daca ecuatia (41) are o radacina complexa $\alpha + i\beta, \ \beta > 0$, atunci si $\alpha - i\beta$ este radacina. Pentru o astfel de pereche gasim doua solutii din sistemul fundamental:

$$y_1 = e^{\alpha x} \cos(\beta x)$$

si

$$y_2 = e^{\alpha x} \sin(\beta x).$$

Repetand rationamentul pentru fiecare radacina r_i, obtinem sistemul fundamental de solutii pentru ecuatia omogena data, format din n functii liniar independente $y_1, ..., y_n$.

Solutia generala va fi combinatia lor liniara cu n constante reale arbitrare C_i, $i = 1, ..., n$.

Exemplu.

Integrati ecuatia

$$y^{''} - 4y^{'} + 5y = 0. \quad \blacksquare$$

Ecuatia caracteristica

$$r^2 - 4r + 5 = 0$$

are radacinile complexe $k = 2 \pm i$.

Deci, un sistem fundamental de solutii este

$$y_1 = e^{2x} \cos x, \;\; y_2 = e^{2x} \sin x.$$

Solutia generala este:

$$y(x) = C_1 e^{2x} \cos x + C_2 e^{2x} \sin x, \qquad C_1, C_2 \in \mathbb{R}.$$

Cazul 4. Daca ecuatia caracteristica are radacina complexa $\alpha \pm i\beta$ cu multiplicitatea m, obtinem $2m$ solutii din sistemul fundamental:

$$\begin{cases} e^{\alpha x} \cos \beta x, \; x e^{\alpha x} \cos \beta x, ..., x^{m-1} e^{\alpha x} \cos(\beta x), \\ \\ e^{\alpha x} \sin \beta x, \; x e^{\alpha x} \sin \beta x, ..., x^{m-1} e^{\alpha x} \sin \beta x. \end{cases} \quad \blacksquare$$

Repetand rationamentul pentru fiecare radacina r_i, obtinem sistemul fundamental de solutii $y_1, ..., y_n$ si, apoi, solutia generala.

Exemplul 1.

Ecuatia diferentiala

$$y^{(4)} + 2y^{''} + y = 0$$

are ecuatia caracteristica

$$r^4 + 2r^2 + 1 = 0,$$

adica

$$(r^2 + 1)^2 = 0.$$

Astfel, $r = \pm i$ sunt radacini complexe duble.

Sistemul fundamental de solutii este:

$$y_1 = \cos\, x,\ y_2 = x \cos\, x,\ y_3 = \sin\, x,\ y_4 = x \sin\, x.$$

Solutia generala

$$y = C_1 \cos\, x + C_2 x \cos\, x + C_3 \sin\, x + C_4 x \sin\, x, \quad C_i \in \mathbb{R}.$$

Exemplul 2.

Integrati ecuatia

$$y^{(4)} - 4y^{'''} + 8y^{''} - 8y^{'} + 4y = 0.$$

$$r^4 - 4r^3 + 8r^2 - 8r + 4 = 0$$

$$(r^2 - 2r + 2)^2 = 0$$

Rezulta ca $r = 1 \pm i$ sunt radacini complexe duble.

$$y_1 = e^x \cos x,\ y_2 = xe^x \cos x,\ y_3 = e^x \sin x,\ y_4 = xe^x \sin x.$$

$$y = C_1 e^x \cos\ x + C_2 x e^x \cos\ x + C_3 e^x \sin\ x + C_4 x e^x \sin\ x, \quad C_i \in \mathbb{R},\ i = \overline{1,4}. \quad \blacksquare$$

Cazul neomogen

$$y(x) = \sum_{i=1}^{n} C_i y_i(x) + y_p(x). \quad \blacksquare$$

Metoda variatiei constantelor:

$$y(x) = C_1(x)y_1(x) + ... + C_n(x)y_n(x).$$

$$\begin{cases} C_1'(x)y_1(x) + C_2'(x)y_2(x) + ... + C_n'(x)y_n(x) = 0, \\ C_1'(x)y_1'(x) + C_2'(x)y_2'(x) + ... + C_n'(x)y_n'(x) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ C_1'(x)y_1^{(n-1)}(x) + C_2'(x)y_2^{(n-1)}(x) + ... + C_n'(x)y_n^{(n-1)}(x) = f(x). \end{cases}$$

$$\begin{vmatrix} y_1(x) & y_2(x) & \ldots\ldots\ldots\ldots\ldots & y_n(x) \\ \\ y_1^{'}(x) & y_2^{'}(x) & \ldots\ldots\ldots\ldots\ldots & y_n^{'}(x) \\ \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \\ y_1^{(n-1)}(x) & y_2^{'}(x) & \ldots\ldots & y_n^{(n-1)}(x) \end{vmatrix},$$

$$C_i^{'}(x) = \varphi_i(x), \quad i = 1, 2, ..., n,$$

unde φ_i sunt functii continue pe (a, b).

$$C_i(x) = \int \varphi_i(x)dx + C_i, \quad C_i \in \mathbb{R}, \quad i = 1, 2, ..., n.$$

$$y(x) = (\int \varphi_1(x)dx + C_1)y_1(x) + (\int \varphi_2(x)dx + C_2)y_2(x) + ... +$$

$$+(\int \varphi_n(x)dx + C_n)y_n(x), \quad C_1, ..., C_n \in \mathbb{R}. \quad \blacksquare$$

Algoritm.

$$a_0 y^{(n)}(x) + a_1 y^{(n-1)}(x) + .. + a_n y(x) = f(x). \tag{42}$$

1) Atasam ecuatia omogena

$$a_0 y^{(n)}(x) + a_1 y^{(n-1)}(x) + .. + a_n y(x) = 0 \tag{43}$$

si aflam un sistem fundamental de solutii $\{y_1(x), y_2(x), ..., y_n(x)\}$.

$$y_{hom}(x) = C_1 y_1 + C_2 y_2 + ... + C_n y_n. \tag{44}$$

2) Cautam solutia ecuatiei neomogene (42) de forma

$$y(x) = C_1(x)y_1(x) + ... + C_n(x)y_n(x), \tag{45}$$

cu functiile $C_1(x), ..., C_n(x)$ determinate din sistemul:

$$\begin{cases} C_1'(x)y_1(x) + C_2'(x)y_2(x) + ... + C_n'(x)y_n(x) = 0, \\ \\ C_1'(x)y_1'(x) + C_2'(x)y_2'(x) + ... + C_n'(x)y_n'(x) = 0 \\ \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \\ C_1'(x)y_1^{(n-1)}(x) + C_2'(x)y_2^{(n-1)}(x) + ... + C_n'(x)y_n^{(n-1)}(x) = f(x). \end{cases} \tag{46}$$

$$C_i'(x) = \varphi_i(x), \quad i = 1, 2, ..., n. \tag{47}$$

$$C_i(x) = \int \varphi_i(x)dx + C_i, \quad C_i \in \mathbb{R}, \quad i = 1, 2, ..., n. \tag{48}$$

3) Solutia generala a ecuatiei date este

$$y(x) = (\int \varphi_1(x)dx + C_1)y_1(x) + (\int \varphi_2(x)dx + C_2)y_2(x) + ...+$$

$$+(\int \varphi_n(x)dx + C_n)y_n(x), \quad C_1, ..., C_n \in \mathbb{R}. \tag{49}$$

4) Daca atasam si o problema Cauchy, determinam cele n constante C_i.

Exemplu.

Integrati ecuatia

$$y'' - y = x^2. \quad \blacksquare$$

$$r^2 - 1 = 0.$$

$$r_1 = 1, \, r_2 = -1.$$

$$y_{hom} = C_1 e^x + C_2 e^{-x}, \quad C_1, C_2 \in \mathbb{R}.$$

Cautam

$$y(x) = C_1(x) e^x + C_2(x) e^{-x}.$$

$$\begin{cases} C_1^{'}(x)e^x + C_2^{'}(x)e^{-x} = 0, \\ \\ C_1^{'}(x)e^x - C_2^{'}(x)e^{-x} = x^2. \end{cases}$$

Obtinem

$$\begin{cases} C_1^{'}(x) = \dfrac{x^2}{2}e^{-x}, \\ \\ C_2^{'}(x) = -\dfrac{x^2}{2}e^{x}. \end{cases}$$

Integrand, obtinem

$$\begin{cases} C_1(x) = (-\dfrac{x^2}{2} - x - 1)e^{-x} + K_1, \\ \\ C_2(x) = (-\dfrac{x^2}{2} + x - 1)e^{x} + K_2, \quad K_1, K_2 \in \mathbb{R}. \end{cases}$$

Deci:

$$y(x) = K_1 e^x + K_2 e^{-x} - x^2 - 2, \quad K_1, K_2 \in \mathbb{R}.$$

Cazuri particulare pentru membrul drept al ecuatiei (42).

Cazul 1.

$$f(x) = A_0 x^s + A_1 x^{s-1} + ... + A_s.$$

Cautam

$$y_p = B_0 x^s + B_1 x^{s-1} + ... + B_s. \tag{50}$$

Daca $a_n = a_{n-1} = ... = a_{n-k+1} = 0$, dar $a_{n-k} \neq 0$, atunci

$$y_p = x^k (B_0 x^s + B_1 x^{s-1} + ... + B_s). \tag{51}$$

Exemplu.

$$y^{''} + y = x^2 + 2x. \quad \blacksquare$$

$$r^2 + 1 = 0$$

$$r = \pm i.$$

$$y_{hom} = C_1 \cos x + C_2 \sin x, \quad C_1, C_2 \in \mathbb{R}.$$

Deoarece $a_2 \neq 0$, cautam o solutie particulara de forma

$$y_p = B_0 x^2 + B_1 x + B_0.$$

Obtinem

$$y_p = x^2 + 2x - 2.$$

Deci, solutia generala este

$$y = C_1 \cos x + C_2 \sin x + x^2 + 2x - 2, \quad C_1, C_2 \in \mathbb{R}.$$

Cazul 2.

$$f(x) = e^{px}(A_0 x^s + A_1 x^{s-1} + ... + A_s),$$

unde p si $A_i,\ i = 0, ..., s$ sunt constante reale.

Daca p nu este radacina a ecuatiei caracteristice, atunci

$$y_p = e^{px}(B_0 x^s + B_1 x^{s-1} + ... + B_s). \tag{52}$$

Daca p este radacina cu multiplicitatea m a ecuatiei caracteristice, atunci

$$y_p = x^m e^{px}(B_0 x^s + B_1 x^{s-1} + ... + B_s). \tag{53}$$

Exemplu.

$$y^{''} + y = e^x(2x+1). \quad \blacksquare$$

$$r^2 + r = 0.$$

$$r = \pm i.$$

$$y_{hom} = C_1 \cos x + C_2 \sin x, \quad C_1, C_2 \in \mathbb{R}.$$

Deoarece 1 nu este radacina caracteristica, vom cauta

$$y_p = e^x(B_0 x + B_1).$$

Rezulta

$$y_p = e^x(x - \frac{1}{2}).$$

Solutia generala:

$$y = C_1 \cos x + C_2 \sin x + e^x(x - \frac{1}{2}), \quad C_1, C_2 \in \mathbb{R}.$$

Cazul 3.

$$f(x) = e^{px}\,[P_0(x) \cos\, qx + Q_0(x) \sin\, qx],$$

unde p si q sunt constante reale, P_0 si Q_0 sunt polinoame in x, cu coeficienti reali.

Daca $p \pm iq$ nu sunt radacini caracteristice, atunci

$$y_p = e^{px}\,[\overline{P}_0(x)\cos\, qx + \overline{Q}_0(x)\sin\, qx], \tag{54}$$

unde $\overline{P}_0(x)$, $\overline{Q}_0(x)$ sunt polinoame in x a.i.

$$\max(\mathrm{grad}(\overline{P}_0(x)),\, \mathrm{grad}(\overline{Q}_0(x))) \leq \max(\mathrm{grad}(P_0(x)),\, \mathrm{grad}(Q_0(x))). \tag{55}$$

Daca $p \pm iq$ sunt radacini caracteristice cu multiplicitatea m, atunci

$$y_p = x^m e^{px}\,[\overline{P}_0(x)\cos\, qx + \overline{Q}_0(x)\sin\, qx]. \tag{56}$$

Exemplu.

$$y^{''} + 4y^{'} + 4y = \cos 2x. \quad \blacksquare$$

Ecuatia caracteristica

$$r^2 + 4r + 4 = 0$$

are radacina reala dubla

$$r = -2.$$

$$y_{hom} = C_1 e^{-2x} + C_2 x e^{-2x}, \quad C_1, C_2 \in \mathbb{R}.$$

Deoarece $\pm 2i$ nu sunt radacini caracteristice, cautam o solutie particulara de forma

$$y_p = A\cos 2x + B\sin 2x.$$

Obtinem

$$y_p = \frac{1}{8}\sin 2x.$$

Deci, solutia generala este

$$y = C_1 e^{-2x} + C_2 x e^{-2x} + \frac{1}{8}\sin 2x, \quad C_1, C_2 \in \mathbb{R}.$$

Algoritm.

$$a_0 y^{(n)}(x) + a_1 y^{(n-1)}(x) + .. + a_n y(x) = f(x). \tag{57}$$

1) Atasam ecuatia omogena

$$a_0 y^{(n)}(x) + a_1 y^{(n-1)}(x) + .. + a_n y(x) = 0. \tag{58}$$

$$y_{hom}(x) = C_1 y_1 + C_2 y_2 + ... + C_n y_n. \tag{59}$$

2) Cautam o solutie particulara $y_p(x)$ a ecuatiei neomogene (57).

3) Solutia generala a ecuatiei neomogene este

$$y(x) = \sum_{i=1}^{n} C_i y_i(x) + y_p(x).$$

4) Daca atasam o problema Cauchy, determinam cele n constante C_i.

Problema 1. Legea lui Newton:

$$m\frac{d^2x}{dt^2} = -kx,$$

unde $k > 0$.

$$\frac{d^2x}{dt^2} + \omega^2 x = 0,$$

cu

$$\omega^2 = \frac{k}{m}.$$

Ecuatia caracteristica este

$$r^2 + \omega^2 = 0,$$

cu radacinile

$$r = \pm\omega i.$$

$$x(t) = C_1 \cos\omega t + C_2 \sin\omega t, \quad C_1, C_2 \in \mathbb{R}.$$

Daca luam

$$C_1 = A\sin\varphi, \;\; C_2 = A\cos\varphi,$$

unde A si φ sunt constante reale arbitrare, avem

$$x(t) = A\sin(\omega t + \varphi).$$

Obtinem, deci, oscilatii armonice cu amplitudinea A si faza initiala φ.

Problema 2. Presupunem ca, pe langa forta elastica, avem si o forta periodica $F = F_0 \cos \lambda t$ si ca rezistenta mediului poate fi neglijata:

$$m\frac{d^2x}{dt^2} = -kx + F_0 \cos \lambda t.$$

$$\omega = \sqrt{\frac{k}{m}},\ a = \frac{F_0}{m}.$$

$$\frac{d^2x}{dt^2} + \omega^2 x = a \cos \lambda t. \tag{60}$$

$$r^2 + \omega^2 = 0$$

$$r = \pm \omega i$$

$$x_{hom}(t) = C_1 \cos \omega t + C_2 \sin \omega t, \quad C_1, C_2 \in \mathbb{R}.$$

Cazul 1. Daca $\lambda \neq \omega$, adica frecventa fortei externe este diferita de frecventa libere, cautam o solutie particulara de forma

$$x_p(t) = A \cos \lambda t + B \sin \lambda t.$$

Rezulta

$$A = \frac{a}{\omega^2 - \lambda^2}, \; B = 0.$$

$$x_p(t) = \frac{a}{\omega^2 - \lambda^2} \cos \lambda t$$

Solutia generala:

$$x(t) = C_1 \cos \omega t + C_2 \sin \omega t + \frac{a}{\omega^2 - \lambda^2} \cos \lambda t, \quad C_1, C_2 \in \mathbb{R}.$$

Daca $\lambda = \omega$, solutia "expodeaza" si este superpozitia a doua oscilatii marginite cu frecvente diferite.

Daca impunem si o conditie initiala:

$$x(0) = 0, \; x^{'}(0) = 0,$$

obtinem

$$C_2 = 0, \; C_1 = -\frac{a}{\omega^2 - \lambda^2}.$$

$$x(t) = \frac{a}{\omega^2 - \lambda^2}(\cos \lambda t - \cos \omega t).$$

$$x(t) = \frac{2a}{\omega^2 - \lambda^2} \sin(\frac{\omega - \lambda}{2} t) \sin(\frac{\omega + \lambda}{2} t).$$

Astfel, solutia se compune din doua frecvente distincte: $(\omega - \lambda)/2$ si $(\omega + \lambda)/2$.

Cazul 2. Daca $\lambda = \omega$, cautam

$$x_p(t) = t(A \cos \omega t + B \sin \omega t),$$

$$A = 0, \ B = \frac{a}{2\omega}.$$

$$x_p(t) = \frac{at}{2\omega} \sin \omega t.$$

$$x(t) = C_1 \cos \omega t + C_2 \sin \omega t + \frac{at}{2\omega} \sin \omega t, \quad C_1, C_2 \in \mathbb{R}.$$

Amplitudinea creste infinit cand t tinde la infinit (rezonanta - fenomen extrem de periculos).

$$x(0) = 0,\ x^{'}(0) = 0$$

$$C_1 = 0,\ C_2 = 0.$$

$$x(t) = \frac{at}{2\omega} \sin \omega t.$$

In realitate, avem frecare, rezistenta aerului, etc.

$$m\frac{d^2x}{dt^2} + \gamma x^{'} + kx = F_0 \cos \lambda t, \tag{61}$$

unde $\gamma > 0$ masoara forta de frecare.

$$mr^2 + \gamma r + k = 0.$$

Cazul 1. Daca
$$\frac{\gamma^2}{4m^2} - \frac{k}{m} < 0,$$
atunci ecuatia caracteristica are radacinile complexe

$$r_{1,2} = -\frac{\gamma}{2m} \pm i\sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}.$$

$$x_{hom}(t) = C_1 e^{-\alpha t}\cos\beta t + C_2 e^{-\alpha t}\sin\beta t, \quad C_1, C_2 \in \mathbb{R},$$

unde
$$\alpha = \frac{\gamma}{2m},\ \beta = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}.$$

O solutie particulara (cu metoda coeficientilor nedeterminati):

$$x_p(t) = \frac{a(\omega^2 - \lambda^2)}{(\omega^2 - \lambda^2)^2 + \gamma_0^2\lambda^2}\cos\lambda t + \frac{\gamma_0 \lambda a}{(\omega^2 - \lambda^2)^2 + \gamma_0^2\lambda^2}\sin\lambda t,$$

unde

$$\gamma_0 = \frac{\gamma}{m},\ \omega = \sqrt{\frac{k}{m}},\ a = \frac{F_0}{m}.$$

Deci,

$$x(t) = C_1 e^{-\alpha t} \cos \beta t + C_2 e^{-\alpha t} \sin \beta t +$$
$$+ \frac{a(\omega^2 - \lambda^2)}{(\omega^2 - \lambda^2)^2 + \gamma_0^2 \lambda^2} \cos \lambda t + \frac{\gamma_0 \lambda a}{(\omega^2 - \lambda^2)^2 + \gamma_0^2 \lambda^2} \sin \lambda t.$$

Cand $t \to \infty$, solutia devine

$$x_\infty = \frac{a}{(\omega^2 - \lambda^2)^2 + \gamma_0^2 \lambda^2} ((\omega^2 - \lambda^2) \cos \lambda t + \gamma_0 \lambda \sin \lambda t).$$

Cazul 2. Daca

$$\frac{\gamma^2}{4m^2} - \frac{k}{m} > 0,$$

atunci ecuatia caracteristica are radacinile

$$r_{1,2} = -\frac{\gamma}{2m} \pm \sqrt{\frac{\gamma^2}{4m^2} - \frac{k}{m}}.$$

$$x_{hom}(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}, \quad C_1, C_2 \in \mathbb{R}.$$

$$x_p(t) = \frac{a(\omega^2 - \lambda^2)}{(\omega^2 - \lambda^2)^2 + \gamma_0^2 \lambda^2} \cos \lambda t + \frac{\gamma_0 \lambda a}{(\omega^2 - \lambda^2)^2 + \gamma_0^2 \lambda^2} \sin \lambda t.$$

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \frac{a(\omega^2 - \lambda^2)}{(\omega^2 - \lambda^2)^2 + \gamma_0^2 \lambda^2} \cos \lambda t + \frac{\gamma_0 \lambda a}{(\omega^2 - \lambda^2)^2 + \gamma_0^2 \lambda^2} \sin \lambda t.$$

Cazul 3. Daca

$$\frac{\gamma^2}{4m^2} - \frac{k}{m} = 0.$$

$$r_{1,2} = -\frac{\gamma}{2m}.$$

$$x_{hom}(t) = e^{-\dfrac{\gamma}{2m}t}(C_1 + C_2 t), \quad C_1, C_2 \in \mathbb{R}.$$

$$x_p(t) = \frac{a(\omega^2 - \lambda^2)}{(\omega^2 - \lambda^2)^2 + \gamma_0^2 \lambda^2} \cos \lambda t + \frac{\gamma_0 \lambda a}{(\omega^2 - \lambda^2)^2 + \gamma_0^2 \lambda^2} \sin \lambda t.$$

$$x(t) = e^{-\dfrac{\gamma}{2m}t}(C_1 + C_2 t) + \frac{a(\omega^2 - \lambda^2)}{(\omega^2 - \lambda^2)^2 + \gamma_0^2 \lambda^2} \cos \lambda t + \frac{\gamma_0 \lambda a}{(\omega^2 - \lambda^2)^2 + \gamma_0^2 \lambda^2} \sin \lambda t.$$

Sisteme de ecuatii diferentiale

$$\begin{cases} \dfrac{dy_1}{dx} = f_1(x, y_1, y_2, ..., y_n), \\ \\ \dfrac{dy_2}{dx} = f_2(x, y_1, y_2, ..., y_n), \\ \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ \\ \dfrac{dy_n}{dx} = f_n(x, y_1, y_2, ..., y_n), \end{cases} \tag{62}$$

unde $f_i : D \subseteq \mathbb{R}^{n+1} \to \mathbb{R},\ f_i \in C^0(D),\ i = 1, 2, ..., n.$

$$y_i(x_0) = y_{i0},\ i = 1, 2, ..., n \tag{63}$$

Daca $Y(x) = (y_1(x), y_2(x), ..., y_n(x))$, atunci

$$\frac{dY}{dx} = F(x, Y), \tag{64}$$

unde $F = (f_1, f_2, ..., f_n)$.

De asemenea,

$$Y(x_0) = Y_0, \tag{65}$$

unde $Y_0 = (y_{10}, y_{20}, ..., y_{n0})$.

Exemplu. Dinamica populatiei.

Sa ne imaginam o insula cu doua specii: iepuri si vulpi (prada si pradator). Rata de variatie a populatiei de un anume tip depinde de marimea populatiei de al doilea tip.

Modelul Lotka-Volterra:

$$\begin{cases} \dfrac{dR}{dt} = aR - \alpha RF, \\ \\ \dfrac{dF}{dt} = -bF + \beta RF, \end{cases}$$

unde $R(t)$ e populatia de iepuri si $F(t)$ e populatia de vulpi.

a si b sunt ratele de crestere ale celor doua tipuri de populatii, iar α si β masoara efectul interactiunii dintre ele.

Sisteme liniare

$$\begin{cases} \dfrac{dy_1}{dx} = a_{11}(x)y_1 + a_{12}(x)y_2 + ... + a_{1n}(x)y_n + f_1(x), \\ \\ \dfrac{dy_2}{dx} = a_{21}(x)y_1 + a_{22}(x)y_2 + ... + a_{2n}(x)y_n + f_2(x), \\ \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \\ \dfrac{dy_n}{dx} = a_{n1}(x)y_1 + a_{n2}(x)y_2 + ... + a_{nn}(x)y_n + f_n(x), \end{cases} \tag{66}$$

unde $a_{ij} : I \subseteq \mathbb{R} \to \mathbb{R},\ a_{ij} \in C^0(I),\ i, j = 1, 2, ..., n$ si
$f_i : I \subseteq \mathbb{R} \to \mathbb{R},\ f_i \in C^0(I),\ i = 1, 2, ..., n.$

$$\frac{dY}{dx} = AY + F, \tag{67}$$

unde

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_n \end{pmatrix},$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots.. & a_{1n} \\ a_{21} & a_{22} & \ldots.. & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots.. & a_{nn} \end{pmatrix}$$

si

$$F = \begin{pmatrix} f_1 \\ f_2 \\ . \\ . \\ . \\ f_n \end{pmatrix}.$$

Fie vectorii $Y_1, Y_2, ..., Y_n$, unde

$$Y_i = \begin{pmatrix} y_{1i} \\ y_{2i} \\ . \\ . \\ . \\ y_{ni} \end{pmatrix}, \tag{68}$$

pentru $i = 1, 2, ..., n$.

Vectorii $Y_1(x)$,...,$Y_n(x)$ s.n. *liniar dependenti* pe (a, b) daca exista n constante $\alpha_1, ..., \alpha_n$, nu toate nule, a.i.

$$\alpha_1(x)Y_1(x) + ... + \alpha_n(x)Y_n(x) \equiv 0, \quad x \in (a, b). \tag{69}$$

Daca relatia (69) este satisfacuta doar pentru $\alpha_1 = ... = \alpha_n = 0$, atunci vectorii $Y_1(x), ..., Y_n(x)$ s.n. *liniar independenti* pe intervalul (a, b). ■

Daca $Y_1(x), Y_2(x), ..., Y_n(x)$ sunt liniar dependenti pe intervalul (a, b), atunci determinantul

$$W(x) \equiv W[Y_1, ..., Y_n] = \begin{vmatrix} y_{11}(x) & y_{12}(x) & \ldots\ldots\ldots\ldots & y_{1n}(x) \\ y_{21}(x) & y_{22}(x) & \ldots\ldots\ldots\ldots & y_{2n}(x) \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots \\ y_{n1}(x) & y_{n2}(x) & \ldots\ldots\ldots\ldots & y_{nn}(x) \end{vmatrix},$$

numit Wronskian, este identic zero pe (a, b). ■

Daca $Y_1, ..., Y_n$ sunt solutii liniar independente ale sistemului liniar omogen asociat sistemului (66), atunci $W(x) = W[Y_1, ..., Y_n] \neq 0$ pentru $x \in (a, b)$. ■

Un ansamblu de n solutii liniar independente ale sistemului liniar omogen asociat sistemului (66) s.n. *sistem fundamental de solutii* ale acestui sistem. ■

$$Y(x) = C_1 Y_1(x) + ... + C_n Y_n(x). \tag{70}$$

Cazul neomogen:

$$Y(x) = \sum_{i=1}^{n} C_i Y_i(x) + Y_p(x). \quad \blacksquare$$

Metoda variatiei constantelor

$$Y(x) = C_1(x) Y_1(x) + ... + C_n(x) Y_n(x).$$

$$\sum_{i=1}^{n} C_i^{'}(x) Y_i + \sum_{i=1}^{n} C_i(x) \frac{dY_i}{dx} = A(\sum_{i=1}^{n} C_i(x) Y_i) + F.$$

$$\begin{cases} \sum\limits_{i=1}^{n} C_i^{'}(x) y_{1i} = f_1(x), \\ \\ \sum\limits_{i=1}^{n} C_i^{'}(x) y_{2i} = f_2(x) \\ \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \\ \sum\limits_{i=1}^{n} C_i^{'}(x) y_{ni} = f_n(x). \end{cases}$$

$$C_i^{'}(x) = \varphi_i(x), \quad i = 1, 2, ..., n,$$

unde φ_i sunt functii continue pe (a, b).

$$C_i(x) = \int \varphi_i(x)dx + C_i, \quad C_i \in \mathbb{R}, \quad i = 1, 2, ..., n.$$

Deci:

$$Y(x) = (\int \varphi_1(x)dx + C_1)Y_1(x) + (\int \varphi_2(x)dx + C_2)Y_2(x) + ...+$$

$$+(\int \varphi_n(x)dx + C_n)Y_n(x), \quad C_1, ..., C_n \in \mathbb{R}. \quad \blacksquare$$

Sisteme liniare cu coeficienti constanti

$$\left\{\begin{aligned} \frac{dy_1}{dx} &= a_{11}y_1 + a_{12}y_2 + ... + a_{1n}y_n + f_1(x), \\ \frac{dy_2}{dx} &= a_{21}y_1 + a_{22}y_2 + ... + a_{2n}y_n + f_2(x), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \frac{dy_n}{dx} &= a_{n1}y_1 + a_{n2}y_2 + ... + a_{nn}y_n + f_n(x), \end{aligned}\right. \tag{71}$$

unde $a_{ij} \in \mathbb{R}$, $i, j = 1, 2, ..., n$ si
$f_i : I \subseteq \mathbb{R} \to \mathbb{R}$, $f_i \in C^0(I)$, $i = 1, 2, ..., n$.

Sistemul omogen atasat:

$$\begin{cases} \dfrac{dy_1}{dx} = a_{11}y_1 + a_{12}y_2 + ... + a_{1n}y_n, \\ \\ \dfrac{dy_2}{dx} = a_{21}y_1 + a_{22}y_2 + ... + a_{2n}y_n, \\ \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ \\ \dfrac{dy_n}{dx} = a_{n1}y_1 + a_{n2}y_2 + ... + a_{nn}y_n. \end{cases}$$

Cautam

$$Y = e^{\lambda x}U,$$

unde $\lambda \in \mathbb{C}$ si

$$U = \begin{pmatrix} u_1 \\ u_2 \\ . \\ . \\ . \\ u_n \end{pmatrix}, \quad U \neq 0.$$

Obtinem

$$(A - \lambda I)U = 0, \tag{72}$$

unde I este matricea unitate.

Astfel, $U \neq 0$ e solutie pentru (72) daca si numai daca

$$\det(A - \lambda I) = 0. \tag{73}$$

Ecuatia (73) este *ecuatia caracteristica* asociata sistemului omogen dat, λ

s.n. *valoare proprie* pentru matricea A si U este un *vector propriu* corespunzator valorii λ.

Multimea valorilor proprii ale matricii A s.n. *spectrul* lui A:

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid \det(A - \lambda I) = 0\}. \tag{74}$$

Pentru orice $\lambda \in \sigma(A)$, vom nota

$$PV_A(\lambda) = \{U \in \mathbb{C}^n \setminus \{0\} \mid (A - \lambda I)U = 0\} \tag{75}$$

multimea tuturor vectorilor proprii corespunzatori valorii proprii λ.

$$\sigma(A) = \{\lambda_1, ..., \lambda_n\}. \tag{76}$$

Cazul 1. Sa presupunem ca toate valorile proprii λ_i, $i = 1, 2, ..., n$, sunt reale si distincte. Pentru fiecare λ_i determinam un vector propriu $U_i \in \mathbb{R}^n$, $U_i \neq 0$.

Vectorii

$$Y_i = e^{\lambda_i x} U_i, \ \ i = 1, 2, ..., n \tag{77}$$

formeaza un sistem fundamental de solutii.

Solutia generala:

$$Y = C_1 Y_1 + ... + C_n Y_n, \quad C_i \in \mathbb{R}, \ \ i = 1, 2, ..., n. \tag{78}$$

Cazul 2. Sa presupunem ca $\lambda = \alpha \pm i\beta$, cu $\beta > 0$, este o valoare proprie complexa a lui A. Determinam $U \in \mathbb{C}^n$, $U \neq 0$. Vectorii

$$Y_1 = \mathrm{Re}(e^{\lambda x} U) \tag{79}$$

si

$$Y_2 = \mathrm{Im}(e^{\lambda x} U) \tag{80}$$

sunt solutii liniar independente ale sistemului omogen dat. Repetand rationamentul pentru toate valorile proprii λ_i, obtinem sistemul fundamental de solutii $\{Y_1, ..., Y_n\}$.

$$Y = C_1 Y_1 + ... + C_n Y_n, \quad C_i \in \mathbb{R}, \; i = 1, 2, ..., n. \qquad (81)$$

Cazul 3. Fie λ v. p. reala cu multiplicitatea $m(\lambda) > 1$. Corespunzator ei, cautam o solutie de forma

$$Y = [P_0 + P_1 x + ... + P_{m(\lambda)-1} x^{m(\lambda)-1}] e^{\lambda x}, \qquad (82)$$

cu $P_0, P_1, .., P_{m(\lambda)-1} \in \mathbb{R}^n$.

$$\begin{cases} (A - \lambda I) P_{m(\lambda)-1} = 0, \\ \\ (A - \lambda I) P_{j-1} = j P_j, \quad j = 1, 2, ..., m(\lambda) - 1. \end{cases} \qquad (83)$$

Astfel,

$$(A - \lambda I)^{m(\lambda)} P_0 = 0. \qquad (84)$$

Putem alege $m(\lambda)$ vectori liniar independenti $P_0^i \in \mathbb{R}^n,\ P_0^i \neq 0$. Determinam apoi P_j^i, for $j = 1, 2, ..., m(\lambda) - 1$. Astfel, obtinem $m(\lambda)$ solutii liniar independente ale sistemului omogen dat. Repetand rationamentul pentru toate v.p. λ, obtinem un sistem fundamental de solutii $\{Y_1, ..., Y_n\}$.

$$Y = C_1 Y_1 + ... + C_n Y_n, \quad C_i \in \mathbb{R},\ i = 1, 2, ..., n. \tag{85}$$

Cazul 4. Fie $\lambda = \alpha \pm i\beta,\ \beta > 0$, o v. p. complexa cu multiplicitatea $m(\lambda) > 1$. Cautam

$$Y = [P_0 + P_1 x + ... + P_{m(\lambda)-1} x^{m(\lambda)-1}] e^{\lambda x}, \tag{86}$$

cu $P_0, P_1, .., P_{m(\lambda)-1} \in \mathbb{C}^n$.

$$\begin{cases} (A-\lambda I)P_{m(\lambda)-1} = 0, \\ \\ (A-\lambda I)P_{j-1} = jP_j, \quad j = 1, 2, ..., m(\lambda)-1. \end{cases} \tag{87}$$

$$(A-\lambda I)^{m(\lambda)} P_0 = 0 \tag{88}$$

si

$$P_j = \frac{1}{j!}(A-\lambda I)^j P_0, \quad j = 1, 2, ..., m(\lambda)-1. \tag{89}$$

Alegem $m(\lambda)$ vectori liniar independenti $P_0^i \in \mathbb{C}^n$, $P_0^i \neq 0$.
Correspunzator lor, determinam P_j^i, $j = 1, 2, ..., m(\lambda) - 1$. Astfel, obtinem $m(\lambda)$ vectori

$$Y_i = [P_0^i + P_1^i x + ... + P_{m(\lambda)-1}^i x^{m(\lambda)-1}]e^{\lambda x},\ i = 1, 2, ..., m(\lambda). \tag{90}$$

Vectorii $\mathrm{Re}\,(Y_i)$ si $\mathrm{Im}\,(Y_i)$ ne dau $2m(\lambda)$ solutii independente ale sistemului dat. Repetand rationamentul pentru toate v.p. λ, obtinem un sistem fundamental de solutii $\{Y_1, ..., Y_n\}$.

$$Y = C_1 Y_1 + ... + C_n Y_n, \quad C_i \in \mathbb{R},\ i = 1, 2, ..., n. \tag{91}$$

Exemplul 1.

Aflati solutia sistemului:

$$\begin{cases} \dfrac{dy_1}{dx} = y_1 + y_2, \\ \\ \dfrac{dy_2}{dx} = 4y_1 + y_2. \quad \blacksquare \end{cases}$$

$$A = \begin{pmatrix} 1 & 1 \\ \\ 4 & 1 \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\lambda_1 = 3,\ \ \lambda_2 = -1.$$

Corespunzator valorii proprii λ_1, determinam un vector propriu $U_1 \neq 0$. Avem:

$$\begin{pmatrix} -2 & 1 \\ & \\ 4 & -\ 2 \end{pmatrix} \begin{pmatrix} u_1 \\ \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}.$$

$$-2u_1 + u_2 = 0.$$

$$U_1 = \begin{pmatrix} 1 \\ \\ 2 \end{pmatrix}.$$

Pentru λ_2, avem

$$\begin{pmatrix} 2 & 1 \\ \\ 4 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}$$

$$U_2 = \begin{pmatrix} 1 \\ \\ -2 \end{pmatrix}.$$

$$Y = C_1 \begin{pmatrix} 1 \\ \\ 2 \end{pmatrix} e^{3x} + C_2 \begin{pmatrix} 1 \\ \\ 2 \end{pmatrix} e^{-x}, \quad C_1, C_2 \in \mathbb{R}.$$

Pe componente:

$$\begin{cases} y_1 = C_1 e^{3x} + C_2 e^{-x}, \\ \\ y_2 = 2C_1 e^{3x} - 2C_2 e^{-x}. \end{cases}$$

Exemplul 2.

$$\begin{cases} \dfrac{dy_1}{dx} = -y_2, \\ \\ \dfrac{dy_2}{dx} = y_1. \quad \blacksquare \end{cases}$$

$$A = \begin{pmatrix} 0 & -1 \\ \\ 1 & 0 \end{pmatrix}$$

Ecuatia caracteristica

$$\det(A - \lambda I) = 0$$

are radacinile

$$\lambda = \pm i.$$

Pentru $\lambda = i$, determinam $U \in \mathbb{C}^2$, $U \neq 0$.

$$\begin{pmatrix} -i & -1 \\ \\ 1 & -i \end{pmatrix} \begin{pmatrix} u_1 \\ \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}.$$

$$u_1 = iu_2.$$

Un vector propriu complex este:

$$U = \begin{pmatrix} 1 \\ \\ -i \end{pmatrix}.$$

Atunci, vectorii

$$Y_1 = \mathrm{Re}(e^{\lambda x} U) \tag{92}$$

si

$$Y_2 = \mathrm{Im}(e^{\lambda x} U), \tag{93}$$

adica

$$Y_1 = \begin{pmatrix} \cos x \\ \\ \sin x \end{pmatrix}, \; Y_2 = \begin{pmatrix} \sin x \\ \\ -\cos x \end{pmatrix},$$

sunt solutii liniar independente ale sistemului dat.

Solutia generala:

$$Y = C_1 \begin{pmatrix} \cos x \\ \\ \sin x \end{pmatrix} + C_2 \begin{pmatrix} \sin x \\ \\ -\cos x \end{pmatrix}, \quad C_1, C_2 \in \mathbb{R}.$$

Pe componente,

$$\begin{cases} y_1 = C_1 \cos x + C_2 \sin x, \\ \\ y_2 = C_1 \sin x - C_2 \cos x. \end{cases}$$

Exemplul 3.

$$\begin{cases} \dfrac{dy_1}{dx} = 4y_1 - y_2, \\ \\ \dfrac{dy_2}{dx} = y_1 + 2y_2. \quad \blacksquare \end{cases} \tag{94}$$

$$\lambda = 3.$$

Cautam

$$Y = [P_0 + P_1 x]e^{\lambda x}, \tag{95}$$

cu $P_0, P_1 \in \mathbb{R}^2$.

$$\begin{cases} (A - \lambda I)P_1 = 0, \\ \\ (A - \lambda I)P_0 = P_1. \end{cases}$$

Astfel,

$$(A - \lambda I)^2 P_0 = 0,$$

adica

$$\begin{pmatrix} 0 & 0 \\ \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}.$$

Putem alege doi vectori liniar independenti $P_0^i \in \mathbb{R}^2,\ P_0^i \neq 0,$

$$P_0^1 = \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix},\ P_0^2 = \begin{pmatrix} 0 \\ \\ 1 \end{pmatrix}.$$

Rezulta

$$P_1^1 = \begin{pmatrix} 1 \\ \\ 1 \end{pmatrix},\ P_1^2 = \begin{pmatrix} -1 \\ \\ -1 \end{pmatrix}.$$

Obtinem, prin urmare, un sistem fundamental de solutii $\{Y_1, Y_2\}$:

$$Y_1 = e^{3x}[\begin{pmatrix} 1 \\ \\ 0 \end{pmatrix} + x(\begin{pmatrix} 1 \\ \\ 1 \end{pmatrix}] = e^{3x}\begin{pmatrix} 1+x \\ \\ x \end{pmatrix},$$

si

$$Y_2 = e^{3x}[\begin{pmatrix} 0 \\ \\ 1 \end{pmatrix} + x\begin{pmatrix} -1 \\ \\ -1 \end{pmatrix}] = e^{3x}\begin{pmatrix} -x \\ \\ 1-x \end{pmatrix}.$$

$$Y = C_1 e^{3x}\begin{pmatrix} 1+x \\ \\ x \end{pmatrix} + C_2 e^{3x}\begin{pmatrix} -x \\ \\ 1-x \end{pmatrix}, \quad C_i \in \mathbb{R},\ i = 1, 2,$$

Pe componente,

$$\begin{cases} y_1 = e^{3x}(C_1 + x(C_1 - C_2)), \\ \\ y_2 = e^{3x}(C_2 + x(C_1 - C_2)). \end{cases}$$

Exemplul 4.

$$\begin{cases} \dfrac{dy_1}{dx} = 2y_1 - y_2 - y_3, \\ \\ \dfrac{dy_2}{dx} = 3y_1 - 2y_2 - 3y_3, \\ \\ \dfrac{dy_3}{dx} = -y_1 + y_2 + 2y_3. \quad \blacksquare \end{cases}$$

$$\lambda_1 = 0,\ \ \lambda_2 = 1,\ m(\lambda_2) = 2.$$

Pentru λ_1, determinam un vector propriu $U_1 \neq 0$.

$$\begin{pmatrix} 2 & -1 & -1 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\begin{cases} u_2 = 3u_1, \\ u_3 = -u_1. \end{cases}$$

$$U_1 = \begin{pmatrix} 1 \\ \\ 3 \\ \\ -1 \end{pmatrix}$$

$$Y_1 = \begin{pmatrix} 1 \\ \\ 3 \\ \\ -1 \end{pmatrix}.$$

Pentru λ_2, cautam solutii de forma

$$Y = [P_0 + P_1 x]e^{\lambda x},$$

cu $P_0, P_1 \in \mathbb{R}^3$.

$$\begin{cases} (A - \lambda I)P_1 = 0, \\ \\ (A - \lambda I)P_0 = P_1. \end{cases}$$

$$(A - \lambda I)^2 P_0 = 0,$$

adica

$$\begin{pmatrix} -1 & 1 & 1 \\ \\ -3 & 3 & 3 \\ \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ \\ u_2 \\ \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \\ \\ 0 \end{pmatrix},$$

de unde

$$u_1 = u_2 + u_3.$$

Putem alege doi vectori liniar independenti $P_0^i \in \mathbb{R}^3,\ P_0^i \neq 0,$

$$P_0^1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ P_0^2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

$$P_1^1 = \begin{pmatrix} 0 \\ \\ 0 \\ \\ 0 \end{pmatrix}, \; P_1^2 = \begin{pmatrix} 0 \\ \\ 0 \\ \\ 0 \end{pmatrix}.$$

Rezulta:

$$Y_2 = e^x \begin{pmatrix} 1 \\ \\ 1 \\ \\ 0 \end{pmatrix}, \quad Y_3 = e^x \begin{pmatrix} 1 \\ \\ 0 \\ \\ 1 \end{pmatrix}.$$

Obtinem astfel in sistem fundamental de solutii $\{Y_1, Y_2, Y_3\}$.

$$Y = C_1 \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + C_2 e^x \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_3 e^x \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad C_i \in \mathbb{R},\ i = 1, 2, 3.$$

Pe componente,

$$\begin{cases} y_1 = C_1 + (C_2 + C_3)e^x, \\ y_2 = 3C_1 + C_2 e^x, \\ y_3 = -C_1 + C_3 e^x. \quad \blacksquare \end{cases}$$

Sistem neomogene

$$Y(x) = \sum_{i=1}^{n} C_i Y_i(x) + Y_p(x). \quad \blacksquare \tag{96}$$

Cazuri particulare pentru membrul drept:

Daca

$$F(x) = \sum_{j=1}^{k} e^{\alpha_j x} (P_j(x) \cos \beta_j x + Q_j(x) \sin \beta_j x), \tag{97}$$

unde $\alpha_j,\ \beta_j \in \mathbb{R}$, $j = 1, 2, ..., k$ si $P_j(x)$, $Q_j(x)$ sunt polinoame in x, atunci

$$Y_p(x) = \sum_{j=1}^{k} e^{\alpha_j x} x^{m_j} (\overline{P}_j(x) \cos \beta_j x + \overline{Q}_j(x) \sin \beta_j x),$$

unde $\overline{P}_j(x)$, $\overline{Q}_j(x)$ sunt polinoame in x cu

$$\max(\mathrm{grad}(\overline{P}_j(x)), \mathrm{grad}(\overline{Q}_j(x))) \leq \max(\mathrm{grad}(P_j(x)), \mathrm{grad}(Q_j(x)))$$

si

$$m_j = \begin{cases} m(\alpha_j + i\beta_j), & \text{daca } \alpha_j + i\beta_j \text{ este v. p. pentru } A, \\ \\ 0, & \text{daca } \alpha_j + i\beta_j \text{ nu este v.p. pentru } A. \end{cases}$$

Exemplul 1.

$$\begin{cases} \dfrac{dy_1}{dx} = -y_2 + x + 1, \\ \\ \dfrac{dy_2}{dx} = y_1 + 2x + 1. \quad \blacksquare \end{cases}$$

Sistemul omogen asociat

$$\begin{cases} \dfrac{dy_1}{dx} = -y_2, \\ \\ \dfrac{dy_2}{dx} = y_1 \end{cases}$$

are solutia

$$\begin{cases} y_1 = C_1 \cos x + C_2 \sin x, \\ \\ y_2 = C_1 \sin x - C_2 \cos x. \end{cases}$$

Cautam

$$\begin{cases} y_1 = C_1 \cos x + C_2 \sin x + ax + b, \\ \\ y_2 = C_1 \sin x - C_2 \cos x + cx + d, \end{cases}$$

Obtinem $a = -2, b = 0, c = 1, d = 3.$

$$\begin{cases} y_1 = C_1 \cos x + C_2 \sin x - 2x, \\ \\ y_2 = C_1 \sin x - C_2 \cos x + x + 3. \end{cases}$$

Exemplul 2.

$$\begin{cases} \dfrac{dy_1}{dx} = -2y_1 + y_2 + e^{-x}, \\ \\ \dfrac{dy_2}{dx} = y_1 - 2y_2 + x. \quad \blacksquare \end{cases}$$

Sistemul omogen asociat

$$\begin{cases} \dfrac{dy_1}{dx} = -2y_1 + y_2, \\ \\ \dfrac{dy_2}{dx} = y_1 - 2y_2 \end{cases}$$

are solutia

$$\begin{cases} y_1 = C_1 e^{-3x} + C_2 e^{-x}, \\ \\ y_2 = -C_1 e^{-3x} + C_2 e^{-x}, \end{cases}$$

unde $C_1, C_2 \in \mathbb{R}$.

Deoarece -1 este v.p. pentru A si a

$$F = \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix} e^{-x} + \begin{pmatrix} 0 \\ \\ 1 \end{pmatrix} x,$$

cautam

$$\begin{cases} y_1 = C_1 e^{-3x} + C_2 e^{-x} + (ax + b)e^{-x} + cx + d, \\ \\ y_2 = -C_1 e^{-3x} + C_2 e^{-x} + (\alpha x + \beta)e^{-x} + \gamma x + \delta. \end{cases}$$

Obtinem

$$\begin{cases} y_1(x) = C_1 e^{-3x} + C_2 e^{-x} + \frac{1}{2} x\, e^{-x} + \frac{1}{3} x - \frac{4}{9}, \\ \\ y_2(x) = -C_1 e^{-3x} + C_2 e^{-x} + (\frac{1}{2} x - \frac{1}{2}) e^{-x} + \frac{2}{3} x - \frac{5}{9}. \quad \blacksquare \end{cases}$$

Metoda variatiei constantelor

Daca $\{Y_1, ..., Y_n\}$ este un sistem fundamental de solutii pentru sistemul omogen asociat, atunci

$$Y(x) = C_1(x)Y_1(x) + ... + C_n(x)Y_n(x),$$

unde $C_1(x), ..., C_n(x)$ urmeaza a fi determinate. Avem:

$$\begin{cases} \sum_{i=1}^{n} C_i^{'}(x) y_{1i} = f_1(x), \\ \\ \sum_{i=1}^{n} C_i^{'}(x) y_{2i} = f_2(x) \\ \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \\ \sum_{i=1}^{n} C_i^{'}(x) y_{ni} = f_n(x). \end{cases}$$

$$C_i^{'}(x) = \varphi_i(x), \quad i = 1, 2, ..., n,$$

unde φ_i sunt functii continue pe (a, b).

$$C_i(x) = \int \varphi_i(x)dx + C_i, \quad C_i \in \mathbb{R}, \quad i = 1, 2, ..., n.$$

Solutia generala

$$Y(x) = (\int \varphi_1(x)dx + C_1)Y_1(x) + (\int \varphi_2(x)dx + C_2)Y_2(x) + ...+$$

$$+(\int \varphi_n(x)dx + C_n)Y_n(x), \quad C_1, ..., C_n \in \mathbb{R}. \quad \blacksquare \tag{98}$$

Algoritm.

Fie sistemul liniar neomogen:

$$\begin{cases} \dfrac{dy_1}{dx} = a_{11}y_1 + a_{12}y_2 + ... + a_{1n}y_n + f_1(x), \\ \\ \dfrac{dy_2}{dx} = a_{21}y_1 + a_{22}y_2 + ... + a_{2n}y_n + f_2(x), \\ \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ \\ \dfrac{dy_n}{dx} = a_{n1}y_1 + a_{n2}y_2 + ... + a_{nn}y_n + f_n(x). \end{cases} \tag{99}$$

1) Atasam sistemul omogen:

$$\begin{cases} \dfrac{dy_1}{dx} = a_{11}y_1 + a_{12}y_2 + ... + a_{1n}y_n, \\ \\ \dfrac{dy_2}{dx} = a_{21}y_1 + a_{22}y_2 + ... + a_{2n}y_n, \\ \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ \\ \dfrac{dy_n}{dx} = a_{n1}y_1 + a_{n2}y_2 + ... + a_{nn}y_n \end{cases} \tag{100}$$

si aflam pentru acesta un sistem fundamental de solutii $\{Y_1(x), Y_2(x), ..., Y_n(x)\}$. Solutia generala a sistemului (100) este

$$Y_{hom}(x) = C_1Y_1 + C_2Y_2 + ... + C_nY_n. \tag{101}$$

2) Cautam solutia generala a sistemului neomogen (99) de forma

$$Y(x) = C_1(x)Y_1(x) + ... + C_n(x)Y_n(x), \tag{102}$$

cu functiile $C_1(x), ..., C_n(x)$ determinate din sistemul:

$$\left\{ \begin{array}{l} \sum\limits_{i=1}^{n} C_i^{'}(x) y_{1i} = f_1(x), \\ \\ \sum\limits_{i=1}^{n} C_i^{'}(x) y_{2i} = f_2(x), \\ \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ \\ \sum\limits_{i=1}^{n} C_i^{'}(x) y_{ni} = f_n(x). \end{array} \right. \tag{103}$$

Acest sistem are o soluti unica

$$C_i^{'}(x) = \varphi_i(x), \quad i = 1, 2, ..., n, \tag{104}$$

unde φ_i sunt functii continue pe (a, b).

$$C_i(x) = \int \varphi_i(x)dx + C_i, \quad C_i \in \mathbb{R}, \quad i = 1, 2, ..., n. \tag{105}$$

3) Solutia generala a sistemului (99) este

$$Y(x) = (\int \varphi_1(x)dx + C_1)Y_1(x) + (\int \varphi_2(x)dx + C_2)Y_2(x) + ...+$$

$$+(\int \varphi_n(x)dx + C_n)Y_n(x), \quad C_1, ..., C_n \in \mathbb{R}. \tag{106}$$

4) Daca atasam o problema Cauchy, putem determina cele n constante C_i. ■

Daca gasim usor o solutie particulara a sistemului neomogen, atunci

$$Y(x) = C_1 Y_1(x) + C_2 Y_2(x) + ... + C_n Y_n(x) + Y_p(x), \quad C_1, ..., C_n \in \mathbb{R}. \quad \blacksquare$$

Exemplul 1.

$$\begin{cases} \dfrac{dy_1}{dx} = -y_2 + \cos x, \\ \\ \dfrac{dy_2}{dx} = y_1 + \sin x. \quad \blacksquare \end{cases}$$

$$\begin{cases} \dfrac{dy_1}{dx} = -y_2, \\ \\ \dfrac{dy_2}{dx} = y_1 \end{cases}$$

$$\begin{cases} y_1 = C_1 \cos x + C_2 \sin x, \\ \\ y_2 = C_1 \sin x - C_2 \cos x, \end{cases}$$

$$\begin{cases} y_1 = C_1(x) \cos x + C_2(x) \sin x, \\ \\ y_2 = C_1(x) \sin x - C_2(x) \cos x. \end{cases}$$

Obtinem

$$\begin{cases} C_1(x) = x + K_1, \\ \\ C_2(x) = K_2, \end{cases}$$

unde $K_1, K_2 \in \mathbb{R}$.

$$\begin{cases} y_1 = K_1 \cos x + K_2 \sin x + x \cos x, \\ \\ y_2 = K_1 \sin x - K_2 \cos x + x \sin x. \end{cases}$$

Exemplul 2.

$$\begin{cases} \dfrac{dy_1}{dx} = -y_1 + y_2 + x, \\ \\ \dfrac{dy_2}{dx} = y_1 - y_2 - x. \\ \\ y_1(0) = y_2(0) = 1. \quad \blacksquare \end{cases}$$

$$\begin{cases} \dfrac{dy_1}{dx} = -y_1 + y_2, \\ \\ \dfrac{dy_2}{dx} = y_1 - y_2 \end{cases}$$

$$\begin{cases} y_1 = C_1 - C_2 e^{-2x}, \\ \\ y_2 = C_1 + C_2 e^{-2x}. \end{cases}$$

$$\begin{cases} y_1 = C_1(x) - C_2(x) e^{-2x}, \\ \\ y_2 = C_1(x) + C_2(x) e^{-2x}. \end{cases}$$

$$\begin{cases} C_1(x) = K_1, \\ \\ C_2(x) = -\dfrac{x}{2}e^{2x} + \dfrac{1}{4}e^{2x} + K_2, \end{cases}$$

unde $K_1, K_2 \in \mathbb{R}$.

$$\begin{cases} y_1 = K_1 - K_2 e^{-2x} + \dfrac{x}{2} - \dfrac{1}{4}, \\ \\ y_2 = K_1 + K_2 e^{-2x} - \dfrac{x}{2} + \dfrac{1}{4}. \end{cases}$$

Din conditia initiala, obtinem $K_1 = 1$ si $K_2 = -1/4$.

$$\begin{cases} y_1 = \dfrac{3}{4} + \dfrac{1}{4} e^{-2x} + \dfrac{x}{2}, \\ \\ y_2 = \dfrac{5}{4} - \dfrac{1}{4} e^{-2x} - \dfrac{x}{2}. \end{cases}$$

Metoda eliminarii

Exemplul 1.

$$\begin{cases} \dfrac{dy_1}{dx} = -4y_1 - 2y_2, \\ \\ \dfrac{dy_2}{dx} = 6y_1 + 3y_2. \quad \blacksquare \end{cases}$$

Derivand in raport cu x in prima ecuatie si inlocuind $y_2^{'}$ din a doua ecuatie, obtinem o singura ecuatie, de ordinul al doilea, pentru y_1:

$$y_1^{''} + y_1^{'} = 0.$$

Rezulta

$$y_1(x) = C_1 + C_2 e^{-x}, \quad C_1, C_2 \in \mathbb{R}.$$

Din prima ecuatie,

$$y_2(x) = -2C_1 - \frac{3}{2} C_2 e^{-x}.$$

Deci:

$$\begin{cases} y_1(x) = C_1 + C_2 e^{-x}, \\ \\ y_2(x) = -2C_1 - \dfrac{3}{2} C_2 e^{-x}. \end{cases}$$

Sisteme neliniare

-dificil de rezolvat

-liniarizare

-metoda eliminarii

-gasirea de combinatii integrabile