1. Să se rezolve sistemul de ecuații liniare

$$\left.\begin{matrix} 2x_1 + x_2 = 4 \\ x_1 + 4x_2 = 9 \end{matrix}\right\}$$

cu ajutorul metodei Jacoby si metodei Gauss-Seidel, efectuand doua iteratii (obtinind ietratiile $x^{(1)}$ si $x^{(2)}$).

Forma initiala a sistemului $Ax = b$: $\left.\begin{matrix} 2x_1 + x_2 = 4 \\ x_1 + 4x_2 = 9 \end{matrix}\right\}$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}; b = \begin{pmatrix} 4 \\ 9 \end{pmatrix}.$$

Pentru rezolvarea sistemului cu metoda Jacoby sau metoda Gauss-Seidel trebuie de trecut la forma convenabila

$$x = Qx + c$$

$$\left.\begin{matrix} 2x_1 = -x_2 + 4 \\ 4x_2 = -x_1 + 9 \end{matrix}\right\} \quad \Rightarrow \quad \begin{cases} x_1 = -0.5x_2 + 2 \\ x_2 = -0.25x_1 + 2.25 \end{cases}$$

$$Q = \begin{pmatrix} 0 & -0.5 \\ -0.25 & 0 \end{pmatrix}; c = \begin{pmatrix} 2 \\ 2.25 \end{pmatrix}$$

Metoda Jacoby

$$\begin{cases} x_1^{(k+1)} = -0.5x_2^{(k)} + 2 \\ x_2^{(k+1)} = -0.25x_1^{(k)} + 2.25, \qquad k = 0,1,2,\dots \end{cases}$$

$$q = \|Q\|_\propto = \left\| \begin{pmatrix} 0 & -0.5 \\ -0.25 & 0 \end{pmatrix} \right\|_\propto = \max(0.5; 0.25) = 0.5 < 1$$

$$k = 0: x^{(0)} = c = \begin{pmatrix} 2 \\ 2.25 \end{pmatrix};$$

$$k = 0: \begin{cases} x_1^{(1)} = -0.5x_2^{(0)} + 2 \\ x_2^{(1)} = -0.25x_1^{(0)} + 2.25, \end{cases} \begin{cases} x_1^{(1)} = -0.5 \cdot 2.25 + 2 = 0.875 \\ x_2^{(1)} = -0.25 \cdot 2 + 2.25 = 1.75, \end{cases}$$

$$x^{(1)} = \begin{pmatrix} 0.875 \\ 1.75 \end{pmatrix};$$

$$\frac{q}{1-q}\left\|x^{(1)} - x^{(0)}\right\|_\infty = 1 * \left\| \begin{pmatrix} 0.875 \\ 1.75 \end{pmatrix} - \begin{pmatrix} 2 \\ 2.25 \end{pmatrix} \right\|_\infty =$$

$$= \left\| \begin{pmatrix} -1.125 \\ -0.5 \end{pmatrix} \right\|_\infty = 1.125 < 0.1$$

$$k = 1: \begin{cases} x_1^{(2)} = -0.5x_2^{(1)} + 2 \\ x_2^{(2)} = -0.25x_1^{(1)} + 2.25, \end{cases} \begin{cases} x_1^{(2)} = -0.5 \cdot 1.75 + 2 = 1.125 \\ x_2^{(2)} = -0.25 \cdot 0.875 + 2.25 = 2.03125, \end{cases}$$

$$x^{(2)} = \begin{pmatrix} 1.125 \\ 2.03125 \end{pmatrix};$$

$$\frac{q}{1-q}\left\|x^{(2)} - x^{(1)}\right\|_\infty = 1 * \left\|\begin{pmatrix} 1.125 \\ 2.03125 \end{pmatrix} - \begin{pmatrix} 0.875 \\ 1.75 \end{pmatrix}\right\|_\infty =$$

$$= \left\|\begin{pmatrix} 0.25 \\ 0.28125 \end{pmatrix}\right\|_\infty = 0.25 < 0.1$$

$$k = 2: \begin{cases} x_1^{(3)} = -0.5x_2^{(2)} + 2 \\ x_2^{(3)} = -0.25x_1^{(2)} + 2.25, \end{cases} \begin{cases} x_1^{(3)} = -0.5 \cdot 2.03125 + 2 = 0.984375 \\ x_2^{(3)} = -0.25 \cdot 1.125 + 2.25 = 1.96875 \end{cases};$$

$$x^{(3)} = \begin{pmatrix} 0.984375 \\ 1.96875 \end{pmatrix};$$

$$\frac{q}{1-q}\left\|x^{(3)} - x^{(2)}\right\|_\infty = 1 * \left\|\begin{pmatrix} 0.984375 \\ 1.96875 \end{pmatrix} - \begin{pmatrix} 1.125 \\ 2.03125 \end{pmatrix}\right\|_\infty =$$

$$= \left\|\begin{pmatrix} -0.140625 \\ -0.0625 \end{pmatrix}\right\|_\infty = 0.140625 < 0.1$$

Metoda Gauss-Seidel

$$\begin{cases} x_1^{(k+1)} = -0.5x_2^{(k)} + 2 \\ x_2^{(k+1)} = -0.25x_1^{(k+1)} + 2.25, \qquad k = 0,1,2,\ldots \end{cases}$$

$$q = \|Q\|_\propto = \left\|\begin{pmatrix} 0 & -0.5 \\ -0.25 & 0 \end{pmatrix}\right\|_\propto = \max(0.5; 0.25) = 0.5 < 1$$

$k = 0$: $x^{(0)} = c = \begin{pmatrix} 2 \\ 2.25 \end{pmatrix}$;

$$k = 0: \begin{cases} x_1^{(1)} = -0.5x_2^{(0)} + 2 \\ x_2^{(1)} = -0.25x_1^{(1)} + 2.25, \end{cases} \begin{cases} x_1^{(1)} = -0.5 \cdot 2.25 + 2 = 0.875 \\ x_2^{(1)} = -0.25 \cdot 0.875 + 2.25 = 2.03125 \end{cases};$$

$$x^{(1)} = \begin{pmatrix} 0.875 \\ 2.03125 \end{pmatrix};$$

$$\frac{q}{1-q}\left\|x^{(1)} - x^{(0)}\right\|_\infty = 1 * \left\|\begin{pmatrix} 0.875 \\ 2.03125 \end{pmatrix} - \begin{pmatrix} 2 \\ 2.25 \end{pmatrix}\right\|_\infty =$$

$$= \left\|\begin{pmatrix} -1.125 \\ -0.21875 \end{pmatrix}\right\|_\infty = 1.125 < 0.1$$

$$k=1:\begin{cases} x_1^{(2)} = -0.5x_2^{(1)} + 2 \\ x_2^{(2)} = -0.25x_1^{(2)} + 2.25, \end{cases} \quad \begin{cases} x_1^{(2)} = -0.5 \cdot 2.03125 + 2 = 0.984375 \\ x_2^{(2)} = -0.25 \cdot 0.984375 + 2.25 = 2.00390625 \end{cases}$$

$$x^{(2)} = \begin{pmatrix} 0.984375 \\ 2.00390625 \end{pmatrix};$$

$$\frac{q}{1-q}\left\|x^{(2)} - x^{(1)}\right\|_\infty = 1 * \left\|\begin{pmatrix} 0.984375 \\ 2.00390625 \end{pmatrix} - \begin{pmatrix} 0.875 \\ 2.03125 \end{pmatrix}\right\|_\infty =$$

$$= \left\|\begin{pmatrix} 0.109375 \\ -0.02734375 \end{pmatrix}\right\|_\infty = 0.109375 < 0.1$$

$$k=2:\begin{cases} x_1^{(3)} = -0.5x_2^{(2)} + 2 \\ x_2^{(3)} = -0.25x_1^{(3)} + 2.25, \end{cases} \quad \begin{cases} x_1^{(3)} = -0.5 \cdot 2.00390625 + 2 = 0.998046875 \\ x_2^{(3)} = -0.25 \cdot 0.998046875 + 2.25 = 2.00048828125 \end{cases}$$

$$x^{(3)} = \begin{pmatrix} 0.998046875 \\ 2.00048828125 \end{pmatrix};$$

$$\frac{q}{1-q}\left\|x^{(2)} - x^{(1)}\right\|_\infty = 1 * \left\|\begin{pmatrix} 0.998046875 \\ 2.00048828125 \end{pmatrix} - \begin{pmatrix} 0.984375 \\ 2.00390625 \end{pmatrix}\right\|_\infty =$$

$$= \left\|\begin{pmatrix} 0.013671875 \\ -0.00341796875 \end{pmatrix}\right\|_\infty = 0.013671875 < 0.1$$